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# Thin Cylindrical Shell of Dust Under Rigid Rotation in General Relativity

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Following the method developed by Papapetrou and Hamoui, a solution of the Einstein equations corresponding to a thin shell of dust under rigid rotation solution is obtained. The region interior to the shell is flat and the exterior vacuum region is chosen as a Lewis manifold. An essential limitation of this type of shell is that it does not allow the limit $M_1$ and $L_1 \to 0$, where $M_1$ and $L_1$ are the mass and the angular momentum per unit length. It is shown that the limitation is a consequence of the fact that the Lewis metric does not contain the Minkowski metric as a special case.

## 1. INTRODUCTION

The known vacuum solutions of Einstein's field equations admit numerous possible structures of the source. Arguments that a possible source of the Kerr solution may be a rotating thin shell were put forward by Newman and Janis (1965), Cohen (1967), and de la Cruz and Israel (1968). In the present work, the analysis of a stationary axisymmetric thin shell is given. The method of Papapetrou and Hamoui (1968) is used to construct the thin shell sources for the stationary axisymmetric fields. By restricting our attentions to space-time admitting group of motions along the axis of symmetry in addition to time and azimuthal angle, a considerable simplicity is achieved in an otherwise very complicated problem.

In Section 2, we shall present a brief review of the Papapetrou–Hamoui method for constructing thin shells in general relativity. In Section 3, we

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have used this technique to obtain the thin shell source for the stationary axisymmetric solution first given by Lewis (1932). In Section 4, we obtain the expressions for the mass and the angular moment per unit length in the case of a rotating dust. In Section 5, we make a discussion of the results obtained. Finally, in Section 6, we add some remarks about those results.

## 2. THE PAPAPETROU–HAMOUI METHOD

Let the 3-cylinder $\Sigma$, defined by the equation

$$\zeta(x^\alpha) = 0 \tag{2.1}$$

be the history of a thin shell. $g^-_{\mu\nu}$ and $g^+_{\mu\nu}$ are the metric tensors in the half-spaces $\zeta < 0$ and $\zeta > 0$, respectively. The coordinate system is chosen such that the metric tensor is continuous through $\Sigma$,

$$[g_{\mu\nu}] \equiv (g^+_{\mu\nu})_{\zeta\to+0} - (g^-_{\mu\nu})_{\zeta\to-0} = 0 \tag{2.2}$$

If $P_\sigma$ is the normal vector to the hypersurface $\Sigma$,

$$P_\sigma = \frac{\partial\zeta}{\partial x^\sigma} \tag{2.3}$$

then the discontinuities of the first derivatives of $g_{\mu\nu}$ are given by

$$[g_{\mu\nu,\sigma}] = B_{\mu\nu}P_\sigma \tag{2.4}$$

where $B_{\mu\nu}(x^\alpha)$ is a symmetric tensor. The energy-momentum tensor is given by

$$T_{\mu\nu} = \tau_{\mu\nu}\delta(\zeta) + \tilde{T}_{\mu\nu} \tag{2.5}$$

where $\tilde{T}_{\mu\nu}$ is an ordinary function of $x^\alpha$ representing a volume distribution of matter. In this paper, we are considering the case in which $\tilde{T}_{\mu\nu}$ vanishes.

The field equations are then given by

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\kappa\tau_{\mu\nu}\delta(\zeta) \tag{2.6}$$

where

$$-2\kappa\tau_{\mu\nu} = B_{\mu\nu}P_\sigma P^\sigma + BP_\mu P_\nu - (B_{\mu\sigma}P_\nu + B_{\nu\sigma}P_\mu)P^\sigma - g_{\mu\nu}(BP_\sigma P^\sigma - \bar{B}) \tag{2.7}$$

with

$$B = g_{\rho\sigma}B^{\rho\sigma}, \qquad \bar{B} = B_{\rho\sigma}P^\rho P^\sigma \tag{2.8}$$

It is evident from (2.7) that $\tau_{\mu\nu}$ is normal to $P^\nu$:

$$\tau_{\mu\nu}P^\nu = 0 \tag{2.9}$$

## 3. STATIONARY CYLINDRICAL SHELL

We consider a thin cylindrical shell of dust under rigid rotation, defined by

$$\zeta \equiv r - a = 0 \tag{3.1}$$

The region interior ($\zeta < 0$) to the shell is flat, and the metric there can be taken as

$$ds_-{}^2 = e^{2\Psi_0}(dr^2 + dz^2) + L_0 r^2\, d\phi^2 - \frac{1}{L_0}\, dt^2 \tag{3.2}$$

where $\Psi_0$ and $L_0 \neq 0$ are dimensionless constants. Since, for the case $e^{2\Psi_0} \neq L_0$, the Riemann tensor is singular at $r = 0$, we shall choose $e^{2\Psi_0} = L_0$, to avoid singularity at the axis of symmetry, as we are interested in the field produced by the surface distribution of matter on the shell alone. (The singularity represents some additional distribution of matter on the axis.) Then, equation (3.2) takes the simple form

$$ds_-{}^2 = L_0(dr^2 + dz^2 + r^2\, d\phi^2) - \frac{1}{L_0}\, dt^2 \tag{3.3}$$

containing only one arbitrary constant, $L_0 > 0$.

The exterior vacuum region $\zeta > 0$ will be chosen as a Lewis manifold (Lewis, 1932) with the metric

$$ds_+{}^2 = e^{2\Psi}(dr^2 + dz^2) + g_{33}^+\, d\phi^2 + 2g_{34}^+\, d\phi\, dt + g_{44}^+\, dt^2 \tag{3.4}$$

where

$$\begin{aligned}
g_{11}^+ &= g_{22}^+ = e^{2\Psi} \equiv k\rho^{(4n^2-1)/2}\\
g_{33}^+ &\equiv \frac{r_0}{2\alpha}(\rho^{2n+1} - \rho^{-2n+1})\\
g_{34}^+ &\equiv -\frac{r_0}{2\alpha}[(\alpha+\beta)\rho^{2n+1} + (\alpha-\beta)\rho^{-2n+1}]\\
g_{44}^+ &\equiv \frac{r_0}{2\alpha}[(\alpha+\beta)^2\rho^{2n+1} - (\alpha-\beta)^2\rho^{-2n+1}]
\end{aligned} \tag{3.5}$$

Here, $k$ and $n$ are dimensionless constants, $r_0$ a constant of dimension of length, $\alpha$ and $\beta$ are constants of dimension of the inverse of length, and

$$\rho \equiv r/r_0 \tag{3.6}$$

This form of the exterior metric was first considered by van Stockum (1937).

Starting with the interior metric (3.3), the continuity conditions (2.2) at $r = a$ yield the following relations:

$$\begin{aligned}
L_0 &= kA^{(4n^2-1)/2}\\
A^{2n} &= (\alpha - \beta)aL_0\\
A^{-2n} &= -(\alpha + \beta)aL_0
\end{aligned} \tag{3.7}$$

where

$$A \equiv a/r_0 \neq 1 \tag{3.8}$$

From (3.7), it follows immediately that

$$(\beta^2 - \alpha^2)a^2L_0^2 = 1 \tag{3.9}$$

Using the conditions (2.4), with

$$P_\sigma = (1, 0, 0, 0) \tag{3.10}$$

we find easily that

$$\begin{aligned} B_{11} &= B_{22} = \tfrac{1}{2}(4n^2 - 1)\frac{L_0}{\alpha} \\ B_{33} &= -\frac{\alpha + 2n\beta}{\alpha} aL_0 \\ B_{34} &= \frac{2n}{\alpha a L_0} \\ B_{44} &= -\frac{\alpha + 2n\beta}{\alpha}\frac{1}{aL_0} \end{aligned} \tag{3.11}$$

With these, we calculate (2.8) immediately,

$$B = \frac{4n^2 - 1}{a}, \qquad \bar{B} = \tfrac{1}{2}(4n^2 - 1)\frac{1}{aL_0} \tag{3.12}$$

and then, from (2.5) and (2.7), we find that the only surviving components of $\tau_{\mu\nu}$ are

$$\begin{aligned} \tau_{33} &= \frac{a}{4\kappa\alpha}[(1 + 4n^2)\alpha + 4n\beta] \\ \tau_{34} &= -\frac{n}{\kappa\alpha a L_0{}^2} \\ \tau_{44} &= \frac{1}{4\kappa\alpha a L_0{}^2}[(3 - 4n^2)\alpha + 4n\beta] \end{aligned} \tag{3.13}$$

## 4. THIN SHELL OF ROTATING DUST

The matter tensor of a rotating dust is, in the general case,

$$T_{\mu\nu} = \rho u_\mu u_\nu \tag{4.1}$$

For a shell of dust rotating about the axis of symmetry, we have only the following nonvanishing components:

$$T_{33} = \rho u_3{}^2, \qquad T_{34} = \rho u_3 u_4, \qquad T_{44} = \rho u_4{}^2 \tag{4.2}$$

Therefore, this case is characterized by the condition

$$T_{33}T_{44} - T_{34}^2 = 0 \tag{4.3}$$

In the case of the thin shell, and using (2.5), (4.3) takes the form

$$\tau_{33}\tau_{44} - \tau_{34}^2 = 0 \tag{4.4}$$

Substituting the values of $\tau_{\mu\nu}$ from (2.7), one obtains

$$(B_{33} - g_{33}B_{11}P^1)(B_{44} - g_{44}B_{11}P^1) - B_{34}^2 = 0 \tag{4.5}$$

Introducing (3.11) into (4.5), we obtain the final condition:

$$4n\frac{\beta}{\alpha} = \left(\frac{4n^2 - 1}{2}\right)^2 - (4n^2 + 1) \tag{4.6}$$

For a given value of the radius $a$ of the shell, we have in the two metrics $ds_-{}^2$ and $ds_+{}^2$ six constants: $k$, $r_0$, $n$, $\alpha$, $\beta$, and $L_0$. The total number of conditions that have to be satisfied is four: the three conditions (3.7) and one condition (4.6). So, we will have at our disposal two arbitrary constants. This is quite reasonable, because the shall can have arbitrary mass and angular momentum per unit length in the direction of the $z$ axis.

To obtain the expression for the mass per unit length, we can use the general form for time-independent systems,

$$M = \int (T^1{}_1 + T^2{}_2 + T^3{}_3 - T^4{}_4)(-g)^{1/2}\, d^3x \tag{4.7}$$

Since, by (2.5) and (2.6),

$$T^\mu{}_\nu = -\frac{1}{\kappa}(R^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu R) \tag{4.8}$$

we find that (4.7) reduces to

$$M = \frac{2}{\kappa}\int R^4{}_4(-g)^{1/2}\, d^3x = \frac{4\pi}{\kappa}\int R^4{}_4(-g)^{1/2}\, dr\, dz \tag{4.9}$$

Now, using the van Stockum field equations (van Stockum, 1937),

$$(-g)^{1/2}R^4{}_4 = \frac{1}{2}\frac{d}{dr}\frac{g_{34}g_{34,r} - g_{33}g_{44,r}}{(g_{34}^2 - g_{33}g_{44})^{1/2}} \tag{4.10}$$

and the Gauss theorem, one obtains immediately from (4.9)

$$M_1 \equiv \frac{dM}{dz} = \frac{2\pi}{\kappa}\frac{\alpha + 2n\beta}{\alpha} \tag{4.11}$$

The angular momentum per unit length can be computed from the general expression valid for the case of axial symmetry,

$$\int T^4{}_\sigma(-g)^{1/2}\xi^\sigma\, d^3x = \text{const} \equiv L \tag{4.12}$$

where

$$\xi^\sigma = (0, 0, 1, 0) \tag{4.13}$$

is the Killing vector describing the symmetry, and $L$ is the angular momentum around the axis of symmetry. Now, for this case, we have

$$(-g)^{1/2}T^4{}_3 = -\frac{1}{\kappa}(-g)^{1/2}R^4{}_3 = -\frac{1}{2\kappa}\frac{d}{dr}\frac{g_{34}g_{33,r} - g_{33}g_{34,r}}{(g_{34}^2 - g_{33}g_{44})^{1/2}} \tag{4.14}$$

Substituting (4.14) in (4.12) and using Gauss' theorem, one finds that

$$L_1 \equiv \frac{dL}{dz} = \frac{2\pi}{\kappa}\frac{n}{\alpha} \tag{4.15}$$

From (4.11) and (4.15), it is evident that the total mass and angular momentum are infinite.

## 5. DISCUSSION

The constants $L_0$ and $k$ entering into (3.3) and the first of (3.5) have to be positive, in order to give the correct signature of the metric. From (3.7), we get

$$\alpha > \beta, \qquad \beta < 0 \tag{5.1}$$

Introducing (4.6) into (4.11), we obtain the following relation for $M_1$:

$$\frac{\kappa}{\pi}M_1 = \tfrac{1}{4}(4n^2 - 1)(4n^2 - 5) \tag{5.2}$$

showing that $M_1$ depends on $n$ only. Therefore, if the quantities $M_1$, $L_1$, and $a$ of the shell are given, we have to proceed as follows. We determine firstly $n$ from (5.2), then $\alpha$ from (4.15) and $\beta$ from (4.6). Finally, we determine $L_0$ from (3.9) and $r_0$ and $k$ from (3.7).

Equation (3.9) can be used for determining $L_0$ only if

$$\beta^2 > \alpha^2 \tag{5.3}$$

Starting from (4.6), we derive the following relation:

$$16n^2(\beta^2 - \alpha^2) = \tfrac{1}{16}(4n^2 - 1)^3(4n^2 - 9)\alpha^2 \tag{5.4}$$

The condition (5.3) is equivalent to

$$4n^2 < 1 \qquad \text{or} \qquad 4n^2 > 9 \tag{5.5}$$

The physical demand $M_1 \geqslant 0$ is then already satisfied.

We also have to demand that the velocity vector $u^\mu$ be timelike or null,

$$g_{\mu\nu}u^\mu u^\nu \leqslant 0$$

Because of (4.1), this is equivalent to

$$T^\mu{}_\mu = g^{\mu\nu}T_{\mu\nu} \leqslant 0 \tag{5.6}$$

since we must have $\rho > 0$. The values of $g^{33}$ and $g^{44}$ on the shell are found from (3.3):

$$g^{33} = 1/L_0 a^2, \qquad g^{44} = -L_0$$

Using (3.13), we find at once

$$T^{\mu}{}_{\mu} = \frac{4n^2 - 1}{2\kappa a L_0} \tag{5.7}$$

Therefore, the demand (5.6) leads to

$$4n^2 - 1 \leqslant 0$$

Combining this result with (5.5), we see that the allowed values of $n$ are only those satisfying the first condition (5.5),

$$4n^2 < 1 \tag{5.8}$$

Using (5.8), we see at once, from (4.6), that

$$n\frac{\beta}{\alpha} < 0$$

According to (5.1), we have $\beta < 0$, and consequently

$$\frac{n}{\alpha} > 0 \tag{5.9}$$

i.e., $n$ and $\alpha$ have the same sign. Since the formula (5.2) does not determine the sign of $n$, we have the two possibilities

$$n = |n|, \qquad \alpha = |\alpha| \qquad \text{or} \qquad n = -|n|, \qquad \alpha = -|\alpha|$$

both giving the same *positive* $L_1$. Now, one can verify directly on (3.5) that changing the sign of both $n$ and $\alpha$ does not change the metric. Consequently, it will be sufficient to consider the one choice of signs only, e.g.,

$$n = |n|, \qquad \alpha = |\alpha| \tag{5.10}$$

Using (5.10), or even (5.9), one derives from the last two relations (3.7) the following result:

$$a > r_0 \tag{5.11}$$

We can determine also the angular velocity $\Omega$ of the motion of the dust particles,

$$\Omega \equiv \frac{d\phi}{dt}$$

The relation

$$u^3 = \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = \Omega u^4 \tag{5.12}$$

combined with (4.1), gives

$$\Omega = \tau^3{}_3/\tau^4{}_3 = \tau^3{}_4/\tau^4{}_4$$

Using the values (3.13) of $\tau_{\mu\nu}$ and the relation (4.6), we obtain

$$\Omega = \frac{\alpha}{16n}(4n^2 - 1)^2 = \frac{16n}{\alpha}\frac{\beta^2 - \alpha^2}{(4n^2 - 1)(4n^2 - 9)} \tag{5.13}$$

these two expressions being identical because of (5.4). From the second expression, we obtain, using (3.9), (4.15), (5.2), and (5.4),

$$L_1 = \frac{\pi}{\kappa}\frac{2n}{\alpha} = \frac{1}{2} M_1 L_0{}^2 a^2 \Omega \frac{4n^2 - 9}{4n^2 - 5} \tag{5.14}$$

This is the generalization for the present case of the classical relation expressing $L_1$ in terms of $M_1$, $a$, and $\Omega$.

With the help of $\Omega$, we can write directly the condition

$$g_{\mu\nu}u^\mu u^\nu \leqslant 0$$

We find

$$g_{\mu\nu}u^\mu u^\nu = (\Omega^2 g_{33} + g_{44})(u^4)^2$$

and consequently we get

$$\Omega^2 g_{33} + g_{44} \leqslant 0$$

Using (3.3), (5.4), and (5.9), a simple calculation leads to

$$\Omega^2 g_{33} + g_{44} = \frac{8}{L_0(4n^2 - 9)}$$

and so we arrive at the condition

$$4n^2 < 9$$

This is different from the condition (5.8) obtained previously. However, the difference is only apparent, since according to the condition (5.5) we have to exclude the values of $n$ in the interval

$$1 < 4n^2 < 9$$

## 6. CONCLUDING REMARKS

The relation (5.9) shows that the shell that has been studied in this paper has $L_1 > 0$. It is easy to see that, in order to obtain also shells with $L_1 < 0$, we have to generalize the Lewis metric by writing the third equation (3.5) in the form

$$g_{34}^{+} = \mp \frac{r_0}{2\alpha}\left[(\alpha + \beta)\rho^{2n+1} + (\alpha - \beta)\rho^{-2n+1}\right]$$

The upper sign in the right-hand side of this relation is leading to $L_1 > 0$, while the lower sign will give $L_1 < 0$.

An essential limitation of the type of shell discussed in this paper is that it does not allow the limit $M_1$ and $L_1 \to 0$. Evidently, a solution having $M_1 = 0 = L_1$ does exist: It is sufficient to assume the Minkowski metric in the two regions $r < a$ and $r > a$. It can be proved that the Lewis metric (3.5) does not contain the Minkowski metric as a special case, and this explains why the limit $M_1$, $L_1 \to 0$ is not allowed for the shell discussed in this paper.

Another aspect of the limitations of the metric (3.5) is that it does not allow the discussion of *static* shells. The reason is very simple. A static shell will be the source of a static gravitational field having $g_{34} = 0$. However, it is easy to show that the metric (3.5) cannot be reduced to a form having

$$g_{34} = 0, \qquad g_{33}g_{44} \neq 0$$

by a choice of the constants $\alpha$, $\beta$, $n$, and $r_0$.

Finally, we notice that according to (5.2) the shell discussed in this paper has an upper limit of the mass $M_1$,

$$0 < M_1 \frac{\kappa}{\pi} \leqslant 5$$

independently of the value of the radius $a$. This situation must be due again to the limitations of the Lewis metric: There are special solutions of the vacuum Einstein equations, containing a smaller number of arbitrary constants than the Lewis solution, which, however, do not constitute special cases of the Lewis solution. It will be interesting to discuss the problem of the cylindrical shell by using as a metric in the region $\zeta > a$ some of these special solutions of the vacuum field equations.

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## REFERENCES

Cohen, J. M. (1967). *Journal of Mathematical Physics*, **8**, 1477.
de la Cruz, V., and Israel, W. (1968). *Physical Review*, **170**, 1187.
Lewis, T. (1932). *Proceedings of the Royal Society A*, **136**, 176.
Newman, E. J., and Janis, A. L. (1965). *Journal of Mathematical Physics*, **6**, 915.
Papapetrou, A., and Hamoui, A. (1968). *Annales de l'Institut Henri Poincaré*, **9**, 2, 179.
van Stockum, W. J. (1937). *Proceedings of the Royal Society, Edinburgh*, **57**, 135.